THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE eighth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Thursday and Friday, December 27 and 28, 1900. There were two sessions each day, opening at 10 o'clock A. M. and 2.30 o'clock P. M. Thirty-eight persons were in attendance among whom were the following twenty-six members of the Society:

Professor Henry Benner, Dr. G. A. Bliss, Professor Oskar Bolza, Professor D. F. Campbell, Professor L. E. Dickson, Professor L. W. Dowling, Dr. J. C. Fields, Professor A. S. Hathaway, Professor Thomas F. Holgate, Dr. Kurt Laves, Professor Heinrich Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor H. B. Newson, Mrs. H. B. Newson, Professor D. A. Rothrock, Dr. F. H. Safford, Professor Oscar Schmiedel, Miss Ida M. Schottenfels, Professor J. B. Shaw, Professor E. B. Skinner, Mr. Burke Smith, Professor E. J. Townsend, Professor C. A. Waldo, Dr. Jacob Westlund, Professor H. S. White.

Professor E. H. Moore, Vice-President of the Society, occupied the chair. The Christmas meeting being the regular time for the election of officers of the section, Professor Thomas F. Holgate was reëlected Secretary and Professors E. J. Townsend and J. B. Shaw were made members of the programme committee.

The following papers were read:

(1) Professor E. H. MOORE: "On the uniformity of continuity."

(2) Professor A. S. HATHAWAY: "Quaternions and fourfold space."

(3) Professor IRVING STRINGHAM: "On the geometry of planes in a parabolic space of four dimensions."

(4) Dr. F. H. SAFFORD: "Flow of heat in two dimensions."

(5) Mr. A. C. LUNN: "Certain mathematical aspects of experimental science."

(6) Mr. E. A. HOOK: "Some properties of circulating decimals."

(7) Professor ARNOLD EMCH: "Note on the congruences of twisted curves."

(8) Professor H. B. NEWSON: "A generalization of the

Wessel-Gauss-Argand diagram" (preliminary communication).

(9) DR. F. R. MOULTON: "On straight line solutions of the problem of n bodies."

(10) DR. G. A. BLISS: "Geodesic lines on an anchor ring."

(11) MR. FRANZ A. LA MOTTE: "On the determination of the algebraic equations invariant under Tschirnhausen transformations, with the parameter representation of all such irreducible equations, with rational coefficients, of the third and fourth degrees."

(12) Professor E. J. TOWNSEND: "Functions of two real variables which are continuous with respect to each variable."

(13) Professor L. E. DICKSON: "The group of the equation for the twenty-seven lines on a general cubic surface."

(14) Professor OSKAR BOLZA: "Concerning the expression of abelian integrals in terms of a fundamental set of integral functions."

(15) Dr. J. C. FIELDS: "Proof of the Riemann-Roch theorem and of the independence of the conditions for adjointness."

(16) Professor OSCAR SCHMIEDEL: "Two reduction formulas applicable to certain particular integrals."

(17) Professor E. B. SKINNER: "Some forms which remain invariant with respect to certain ternary monomial substitution groups."

(18) Professor JAMES B. SHAW: "Note indicating a new development of a determinant."

(19) Professor E. H. MOORE: "On double limits."

(20) Professor E. H. MOORE: "Concerning the Harnack theory of improper definite integrals."

(21) Professor L. E. DICKSON: "Canonical forms of quaternary abelian substitutions in an arbitrary Galois field."

(22) Miss IDA M. SCHOTTENFELS: "Proof of the existence of a particular substitution group of degree twenty-one and order 20160."

Professor Hathaway also read a paper on "Pure mathematics for engineering students," which was followed by a very interesting discussion. This paper is printed in the present number of the BULLETIN.

Mr. Lunn was introduced to the Society by Dr. Moulton, Mr. Hook by Professor Skinner, Mr. La Motte by Professor Moore. Professor Stringham's paper was presented to the Society through Professor Moore and in the author's absence

was read by Professor Hathaway; Professor Emch's paper and Professor Dickson's paper (No. 21) were read by title. Abstracts of the papers follow below.

In Professor Moore's paper on the uniformity of continuity, the well known theorem and a generalization of it which occurs in the theory of discontinuous functions are proved by a very simple method involving the partition of a fundamental interval (a, b) into three halves (a, d), (c, e), (d, b), where the points $a\ c\ d\ e\ b$ are the extremities of the four quarters of (a, b).

The first part of Professor Hathaway's paper consisted of a development of quaternions from the definition: Quaternions is the most general linear associative algebra in which division is determinate. The merit claimed is in starting from this known property as a definition, and developing smoothly and briefly a working knowledge of quaternions, without reference to the units $1, i, j, k$.

The characteristic equation of a number p is the equation of least degree with numerical coefficients that is satisfied by p, say,

$$p^{r+1} + c_1 p^r + \cdots + c_r p = 0.$$

There is one and only one such equation for a given number; it is not factorable into expressions of the same form; and c_r is not zero. This determines a unit

$$I = -(p^r + c_1 p^{r-1} + \cdots + c_{r-1} p)/c_r,$$

with the properties

$$I^2 = I, \quad Iq = qI = q,$$

where q is any quaternion. I is the scalar unit, and xI, or simply x, where x is a numerical coefficient, will denote a scalar number. A scalar number may also be defined as a quaternion whose square is a positive scalar; and a vector number is defined as a quaternion whose square is a negative scalar. Scalars are seen to be commutative factors.

Using scalar coefficients, the characteristic equation becomes, in irreducible form, $p^r + c_1 p^{r-1} + \cdots + c_r = 0$; and as this factors in the same way as ordinary equations with *real* coefficients, we must have $r = 1$ or $r = 2$, according as p is a scalar or a non-scalar. When p is a scalar, we take the square of its characteristic, and thus have a character-

istic equation for any number p, of the form $(p-p_1)^2+p_2^2=0$, where p_1, p_2 are scalars depending upon p. We define:

Scalar of $p = Sp = p_1$; vector of $p = Vp = p - p_1 = p - Sp$.

Tensor of $p = Tp = \sqrt{(p_1^2+p_2^2)}$; unit or versor of $p = Up = p/Tp$.

Conjugate of $p = Kp = Sp - Vp$; angle of $p = \angle p = \cos^{-1} SUp$.

The characteristic equation of p is, in this notation,

$$(a) \qquad p^2 - 2Sp \cdot p + Tp^2 = 0.$$

Other forms of the same characteristic are

$$Tp^2 = pKp = Kp \cdot p = Sp^2 - Vp^2 = Sp^2 + TVp^2,$$

$$p^{-1} = Kp/Tp^2, \text{ etc.}$$

The uniqueness of the characteristic (a), determines the uniqueness of the resolutions

$$p = Sp + Vp = Tp \cdot Up = Up \cdot Tp, \text{ etc.}$$

From the characteristic equation of pq, we find by operating with $p^{-1}(\ )p$, the characteristic of qp, and hence, $T(pq) = T(qp)$, $S(pq) = S(qp)$. In particular, if α, β be any two vectors we find

$$(\alpha\beta + \beta\alpha)(\alpha\beta - \beta\alpha) = (\alpha\beta)^2 - (\beta\alpha)^2 = 2S\alpha\beta(\alpha\beta - \beta\alpha);$$

and thence, whether $\alpha\beta = \beta\alpha$ or not, we find $\alpha\beta + \beta\alpha = 2S\alpha\beta$. Thus $(\alpha+\beta)^2$ is a scalar, whence $\alpha+\beta$ is either zero or a vector. Thence follow the usual distributive properties of the functional symbols S, V, K over a sum. Also, the conjugate of a product equals the product of the conjugates of its factors in reverse order; the tensor of a product equals the product of the tensors of its factors, etc. The usual relations of products of vectors are found as in Tait, §§ 89–91; and in particular

$$\delta \cdot S(\alpha\beta\gamma) = \alpha S(\beta\gamma\delta) + \beta S(\gamma\alpha\delta) + \gamma S(\alpha\beta\delta);$$

or, given three independent vectors, any fourth vector can be expressed in terms of them. If there is only one independent vector, the system is the ordinary imaginary system. If there are two unit vectors α, β that are independent, then $\alpha-\beta$, $\alpha+\beta$, $(\alpha-\beta)(\alpha+\beta) = 2V\alpha\beta$ are three independent vectors, whose units are, say i, j, k, with the relations $i^2 = j^2 = k^2 = -1 = ijk$.

The second part of the paper is an application of quaternions to fourfold space, first given in the BULLETIN, November, 1897, and afterwards extended in the papers on Alternate processes (*Proceedings of the Indiana Academy of Sciences*. 1897) and Linear transformations in four dimensions (abstract in BULLETIN, November, 1898, no. 5, p. 93). The application is founded on the definitions

(1) Line $(w + xi + yj + zk)$ = line whose components along the four mutually perpendicular axes of reference are w, x, y, z;

(2) Line p + line q = line $(p + q)$,

(3) Line $p \cdot$ line q = line pq.

With p as multiplier and q as multiplicand there are two products pq and qp. Such multiplications *with* or *by* p, are the two kinds of multiplication referred to as direct and contra multiplication in the first paper, and are now called in-multiplication and by-multiplication. A given in- or by-multiplication is shown to possess a system of invariant planes, one and only one through each line; and the angular displacement in each plane is constant in magnitude and therefore in sense, and equal to the angle of the multiplier. The ratio of elongation is the tensor of the multiplier. The invariant planes of an in-multiplication are called in-parallel planes; and by-parallel planes are invariant planes of a by-multiplication. A given angularly-directed plane is shown to be a plane of an in-parallel system of one and only one unit vector α, and a plane of a by-parallel system of one and only one unit vector β; and these vector units (corresponding to directions in ordinary or vector space) are called the in direction and by-direction of the given plane. In the case of two planes, the angle between these in-directions is called their in-angle of inclination, and the angle between their by-directions their by-angle of inclination. It is shown that these angles remain unaltered by rigid displacement, the effect of such rigid displacement being to give all in-directions of planes a revolution about one vector axis, and all by-directions a revolution about another vector axis. It is shown that two planes have in general two and only two common perpendiculars (a perpendicular to a plane being a plane that "transverses" it in a right diedral angle). These two common perpendiculars are "orthogonal" to each other, *i. e.*, every line of one is perpendicular to every line of the other. In-parallel or by-parallel planes are shown to have a continuous turn-paral-

lel system of orthogonal common perpendiculars, and all "plane" angles between the two lines of intersection with a common perpendicular are equal. For any two planes the two plane angles of intersection by the orthogonal common perpendiculars are the half sum and half difference of the in-angle and by-angle between the planes. The area-projecting factor between two planes is in sign and magnitude the product of the cosines of their plane angles. The second part of this paper will appear in the *Transactions*.

Professor Stringham adopts essentially definitions (1) and (2) of Professor Hathaway's paper, but not definition (3). The subject is analytically treated by equations of loci, with the interpretation and manipulation of their parameters. The fundamental equation of a plane through the origin is $\alpha r + r\beta = 0$ where α, β are unit vectors and r is the director line of any point R of the plane. This is the plane which Professor Hathaway fixes as of in-direction $\pm \alpha$ and by-direction $\mp \beta$. After developing the analytical material of distances between points, and points and planes, and of angles between lines, the author discusses the maximum angle between two variable lines lying in given planes, applying calculus methods to the varying parameters. He finds in general two such angles called the plane angles of the two planes. He finds the projective factor between the two planes $\{\alpha, \beta\}$, $\{\alpha', \beta'\}$ to be $-\frac{1}{2}(S\alpha\alpha' + S\beta\beta')$, which is also the product of the cosines of the plane angles. This expression is Professor Hathaway's

$$\tfrac{1}{2}(\cos\theta + \cos\varphi) = \cos\tfrac{1}{2}(\theta + \varphi)\cos\tfrac{1}{2}(\theta - \varphi).$$

Professor Stringham defines this projective factor as the cosine of the angle between the planes. This paper is to be published in the *Transactions*.

Dr. Safford inserts a term expressing surface leakage in the usual differential equation for the flow of heat in two dimensions, considers the possibility of obtaining solutions of a particular form for this equation, and deduces the isothermal curves and curves of flow. The methods used involve the discussion of a solution of a partial differential equation in the form λ, R_1, R_2, where R_1 and R_2 are functions of the two independent variables respectively, and are the general solutions of two ordinary differential equations. The factor λ is a function of both variables but does not contain the integration constants appearing in R_1 and R_2.

Mr. Lunn showed that if a physical hypothesis is given by implication in the form of an equation defining the value of a variable quantity as a function of the time, there exists at least a double infinity of functions which are not distinguishable from it by measurement, and so far to be considered as equally verified by observation. Also if there exists one satisfactory hypothesis, there exists an infinity. Hence an induction using only the observed values and no hypothetical elements can never give a definite result.

Mr. Hook gave a method by which the number of digits in the repetend of any circulating decimal may be found when the prime factors of the denominator are known. His result differs from similar results found in text books in that it gives the exact number of digits while most such formulas yield only a number of which this is a divisor.

In Professor Emch's paper a study is made of the congruence of curves of intersection of the surfaces $F(x, y, z, a, b) = 0$ and $\varphi(x, y, z, a, b) = 0$. If $b = f(a)$ the system of curves lies upon a surface which has an envelope when

$$\frac{\partial F}{\partial a} + \frac{\partial F}{\partial b} \cdot \frac{db}{da} = 0 \text{ and } \frac{\partial \varphi}{\partial a} + \frac{\partial \varphi}{\partial b} \cdot \frac{db}{da} = 0.$$

Eliminating x, y, z from these conditions and the equations of the surfaces, a new relation

$$\psi\left(a, b, \frac{db}{da}\right) = 0$$

is found. If this last equation has a singular solution, the corresponding system of curves forms the singular surface of the congruence and all surfaces of the congruence are tangent to it.

In Dr. Moulton's paper those solutions of the problem of n bodies are sought in which the bodies always lie in a straight line and describe conic sections with respect to their center of gravity. The problem was solved by Lagrange for three bodies; this paper reduces the problem in the general case of n bodies to the solution of simultaneous algebraic equations. The solutions, which can be obtained in each case for a particular value of one of the masses, are studied as functions of this mass regarded as variable; it is shown that these functions have no poles and, by the non-

vanishing of the Jacobian of the functions with respect to the coördinates, that they have no branch points. Therefore there are $\frac{1}{2}n!$ distinct straight line solutions of the problem of n bodies for all values of the masses. The method applies with only slight modifications for laws of force varying as any power of the distance.

Dr. Bliss discussed the forms of the geodesic lines on the anchor ring. The curves of minimum length were found by the methods of the calculus of variations and the resulting equations, which define a doubly infinite system of geodesic lines, were expressed in terms of Weierstrassian σ, ζ, and $\wp$ functions. By applying the properties of these functions the shapes of the geodesic lines are determined. In the paper all points of the surface of this ring are classified. The paper is intended for publication in the *Transactions*.

Mr. La Motte's paper is in abstract as follows: Given a realm of rationality Ω and in Ω an equation $f(x) = 0$ of the nth degree. This latter is invariant under Tschirnhausen transformations of the same realm if, and only if, its Galois group in Ω is a subgroup of one or more of certain groups which as to type are fully determined for each degree n. By means of resolvents a general method is developed to find parameter representations of all *irreducible* equations whose Galois groups are subgroups of one of the groups mentioned. In the cases of the third and fourth degrees these parameter representations are actually found. They represent for every set of rational parameters a required equation of the given degree, and also every required equation is represented by one of them and a set of rational parameters. These formulæ for the fourth degree are

$$x^4 + px^3 + qx^2 + rx + s =$$

$$(1) \qquad x^4 + \frac{4\vartheta(1-\xi) + \eta^2}{1 - \xi + \varepsilon(\eta - \varepsilon\vartheta)}\,(\varepsilon x^3 + \xi x^2 + \eta x + \vartheta) = 0$$

and

$$(2) \qquad x^4 + \frac{4p_2 - p_3^2}{p_1}\left(x^3 + \frac{p_2}{p_1}x^2 + \frac{p_3}{p_1}x + \frac{1}{p_1}\right) = 0,$$

with the added negative conditions 1° that the discriminant of

$$(3) \qquad t^3 - qt^2 + (pr - 4s)t - (r^2 - 4qs + p^2s) = 0$$

does not vanish, and 2° that none of the three numbers

$$\sqrt{p^2 - 4q + 4\sigma_i} \qquad (i = 1, 2, 3)$$

is rational, where σ_i $(i = 1, 2, 3)$ are the roots of (3). For the third degree we have the simple formula

$$x^3 + qx + r = x^3 - \frac{1 + 27\varepsilon^3}{4\delta^2}\left(x + \frac{\varepsilon}{\delta}\right) = 0,$$

with the negative condition that

$$4q^3 + 27r^2 \neq 0.$$

In these formulas the Greek letters denote parameters which are, as the case may be, arbitrary or definite quantities of the realm Ω.

Professor Townsend's paper dealt with the applications of the double limit to the investigation of certain properties of a function defined as follows: Let $f(x, y)$ be a one-valued function of two real independent variables x and y and continuous with respect to each variable separately within the region, $\alpha \leqq x \leqq \beta$, $y_0 < y \leqq y_n$. Then it follows, that within this region the regular points (*i. e.*, points where $L_{\substack{x=a\\y=b}} f(x, y) = f(a, b)$) must be everywhere dense, but at the same time the irregular points (*i. e.*, where either $L_{\substack{x=a\\y=b}} f(x, y)$ does not exist or is different from $f(a, b)$) may be also everywhere dense. When further $f(x, y)$ is defined on the boundary $y = y_0$ for each value of $x = x_0$ by the limit $L_{y=y_0} f(x_0, y) = f(x_0, y_0)$, then it follows that $f(x, y_0)$ can be discontinuous in x at a set of points everywhere dense in the interval (α, β), but not at every point of this interval. By aid of this result it was shown that the points on the boundary $y = y_0$ at which $f(x, y)$ is continuous with respect to both variables together must lie everywhere dense. When all of the points of the boundary, including the end points of the interval, are regular points, then we have the necessary and sufficient condition for the uniform convergence of the function $f(x, y)$ toward the boundary function $f(x, y_0)$. These same results hold concerning the convergence of an infinite series of continuous functions of x, since the convergence of such a series is but a special case of the convergence of the function defined above toward the boundary

function $f(x, y_0)$. The necessary and sufficient condition that $f(x, y_0)$ and hence that a function defined by an infinite series of continuous functions be continuous at $x = x_0$ is that there exists a set of values of y, say $y = a_1, a_2, a_3, \cdots, a_k, \cdots$, dense at y_0, such that for each such value of a_k we have $|f(x, y_0) - f(x, a_k)| < \sigma$ for every value of x within a definitely defined interval $(x_0 - \delta_{a_k}, x_0 + \delta_{a_k}')$, which may, however, vary with a_k and may have, in fact, an under limit equal to zero, thus differing from the necessary and sufficient condition for uniform convergence. The full paper from which this is taken constituted the author's dissertation for the doctor's degree and has already been published.

Professor Dickson's paper (No. 13), outlined a chapter of his work on linear groups which is soon to appear in Teubner's Sammlung. An orthogonal group O of order 25920 on 5 indices modulo 3 is readily shown to be isomorphic with the abelian linear group for the tri-section of the periods of a hyperelliptic function of four periods. A rectangular table for O having 27 rows serves to define an isomorphic substitution group which is recognized as a subgroup of index two of the group of the equation for the 27 lines on a general cubic surface. This proof of the identity of the two problems avoids the lengthy calculations of M. Jordan, the discoverer of the relation.

The object of Professor Bolza's paper* is to prove by methods of the theory of functions the converse of the propositions given in Baker's work on abelian functions, chapter IV, concerning the expression of abelian integrals in terms of a fundamental set of integral functions.

Let $f(x, y) = 0$ be an irreducible equation of degree n in y; suppose that for a finite value $x = a$ there are m branch places $(a, b_1), (a, b_2), \cdots, (a, b_m)$ of orders $w_1, w_2, \cdots, w_m$ respectively, where $(w_1 + 1) + (w_2 + 1) + \cdots + (w_m + 1) = n$, and put, in the vicinity of (a, b_k), $x - a = t_k^{w_k - 1}$.

(a) If then $g_0, g_1, \cdots, g_{n-1}$ constitute a fundamental set of integral functions for $f(x, y) = 0$, and if the expansion of g_i in (a, b_k) is

$$g_i = \sum_{\nu=0}^{\infty} c_{k\nu}^{(i)} t_k^{\nu}$$

the determinant

* After the paper was finished, I found that I had been anticipated in essential points, by Landsberg, Zur Algebra des Riemann-Roch'schen Satzes, *Math. Annalen*, vol. 50, p. 333. O. BOLZA.

$$D = |c_{k\nu}^{(0)} c_{k\nu}^{(1)} \cdots c_{k\nu}^{(n-1)}|, \qquad (\nu = 0, \cdots, w_k; k = 1, 2, \cdots, m),$$

is different from zero.

(*b*) Let now $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ denote the complementary functions associated with $g_0, g_1, \cdots, g_{n-1}$; then it follows from the characteristic property* of the complementary functions, that in the vicinity of (a, b_j)

$$\sum_{i=0}^{n-1} \sum_{\mu=0}^{\infty} c_{k,\,\rho+\mu(w_k+1)}^{(i)} (x-a)^{\mu} \gamma_i = \frac{\delta_{jk}}{(w_j+1) t_j^{\rho}}$$

where $\delta_{jk} = 1$ or 0 according as $j = k$ or $+k$, for $\rho = 0, 1, \cdots, w_k$; $k = 1, 2, \cdots, n$. The determinant of these n equations with $\gamma_0, \gamma_1, \cdots, \gamma_{n-1}$ as unknown quantities is an ordinary power series in $(x - a)$ which for $x = a$ reduces to the non-vanishing determinant D. Thence follows

1. In the vicinity of (a, b_j)

$$\gamma_i = \frac{P(t_j)}{t_j^{w_j}} \qquad (j = 1, 2, \cdots, m),$$

P being the general symbol for an ordinary power series.

2. If $\nu = \rho + q(w_k + 1)$, where $0 \leqq \rho \leqq w_k$, and if we define

$$\gamma_{k\nu} = \sum_{\mu=0}^{q} \sum_{i=0}^{n-1} c_{k,\,\rho+\mu(w_k+1)}^{(i)} (x-a)^{\mu} \gamma_i,$$

this function γ_k has in the vicinity of (a, b_j) the expansion

$$\frac{\gamma_{k\nu}}{(x-a)^q} = \frac{\delta_{jk} t_j^{-\nu}}{w_j + 1} + t_j P(t_j).$$

(*c*) If, in particular, $g_0 = 1, g_1, \cdots, g_{n-1}$ form a *normal* fundamental set ("normal basis") of dimensions $d_0 = 1, d_1, \cdots, d_{n-1}$ respectively, the expansions of the functions γ_i and $\gamma_{k\nu}$ at infinity can be obtained by combining the above results with Weber's theorem according to which in this case

$$h_0 = \frac{g_0}{x^{d_0}}, \; h_1 = \frac{g_1}{x^{d_1}}, \cdots, h_{n-1} = \frac{g_{n-1}}{x^{d_{n-1}}}$$

constitute a normal fundamental set for the algebraic equation

$$f\left(\frac{1}{z}, y\right) = 0.$$

* Baker, l. c., p. 63.

The result is that

1. The integrals

$$\int x^{d_i-2}\gamma_i\,dx \qquad (i=1, 2, \cdots, n-1)$$

are integrals of the first kind.

2. The integral

$$\int \frac{-\nu\gamma_{k\nu}\,dx}{(x-a)^{\nu+1}}$$

is an integral of the second kind with the one pole (a, b_k) and the expansion

$$\frac{1}{t_k^{\nu}}+P(t_k)$$

in its vicinity.

3. The integral

$$\int\left[\frac{\sum_{i=0}^{n-1}g_i(\xi,\eta)\gamma_i(x,y)}{x-\xi}-\frac{\sum_{i=0}^{n-1}g_i(\xi',\eta')\gamma_i(x,y)}{x-\xi'}\right]dx$$

is an elementary integral of the third kind with the logarithmic points (ξ, η) and (ξ', η').

Dr. Field's proof of the Riemann-Roch theorem is as follows: Let $F(z, u)=0$ be the equation to an irreducible algebraic curve of degree n. The form of this equation is supposed to be such that the only multiple points are double points which are not at the same time branch points. Regard u as the dependent variable and further assume that no line parallel to the axis of u passes through more than one double point, or is tangent at a double point, or is an asymptote, and also that the asymptotes are all distinct from one another and no two parallel to each other. To an equation of the character in question, the equation of any irreducible algebraic curve can be reduced by a birational transformation.

By $(a_1, b_1), \cdots, (a_d, b_d)$ indicate the d double points, and by $(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$ any other q points of the curve, and consider the form of the function

$$(1)\qquad \sum_{\lambda=1}^{d+q}\frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)}+T(z, u),$$

where $T(z, u)$ is an arbitrary polynomial in (z, u) and the quantities γ_λ are arbitrary constants. This form represents the most general rational function of (z, u) which, apart from infinities at ∞, only becomes infinite for points among the q points

$$(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$$

and for these to the first order only.

By a purely algebraic process seek to determine the limitations which must be imposed upon the form (1) in order that it may represent a function which is finite at ∞. We find that the most general rational function whose infinities are all included under the q infinities here in question is given by the form

$$(2) \qquad \sum_{\lambda=1}^{d+q} \frac{\gamma_\lambda F(a_\lambda, u)}{(z-a_\lambda)(u-b_\lambda)} + c,$$

where c is an arbitrary constant and the coefficients γ_λ satisfy the system of $\frac{1}{2}(n-1)(n-2)$ equations

$$(3) \qquad \sum_{\lambda=1}^{d+q} \gamma_\lambda a_\lambda^i b_\lambda^k = 0, \qquad (i+k=0, 1, 2, \cdots, n-3).$$

In the case $q=0$ the form (1), for arbitrary values of the coefficients γ_λ, represents the most general rational function of (z, u) which becomes infinite only at ∞, and the form (2), for conditioned values of the coefficients γ_λ furnished by the equations (3), represents the most general rational function which is nowhere infinite.

Now, an algebraic function which is nowhere infinite can only be a constant, and in the case in question—namely $q=0$—it follows that the coefficients γ_λ in (2) must all have the value 0. The system of equations

$$(4) \qquad \sum_{\lambda=1}^{d} \gamma_\lambda a_\lambda^i b_\lambda^k, \quad (i+k=0, 1, 2, \cdots, n-3)$$

can therefore only be satisfied when the quantities γ_λ all have the value 0.

From the interpretation of the equations (4), it immediately follows that the d conditions for adjointness in the case of a curve of degree $n-3$, and therefore also in the case of a curve of higher degree. are independent of one another.

In the general case $q \neq 0$, an examination of the system of equations (3) shows that the number of arbitrary quantities γ_λ involved in their solutions is $q - s$, where s is the "*strength*" of the system of q points $(a_{d+1}, b_{d+1}), \cdots, (a_{d+q}, b_{d+q})$ in determining an adjoint polynomial of degree $n - 3$. From (2) it is seen that $q - s + 1$ is the number of arbitrary constants involved in the expression of the most general rational function of (z, u), whose infinities, all of the first order, correspond to points among the q points here in question.

Professor Schmiedel presented the results of an investigation into the general formula of reduction for the function

$$\int x^m y^n dx, \quad \text{where} \quad y = \sum_0^i t a_t x^t,$$

and deduced two formulas by which the reduction of the exponents m and n may be effected.

The forms studied by Professor Skinner are such that they are invariant with respect to substitutions of the form $z_1' = a_i z_j$, $z_2' = a_j z_j$, $z_3' = a_k z_k$, $(i\ j\ k = 1, 2, 3$ in some order), where a_1, a_2, a_3 are roots of unity and $a_1 a_2 a_3 = \pm 1$. The groups for which i, j, k have the order 1, 2, 3 are abelian and all such ternary monomial groups can be generated by at most two independent generators. If T_1, T_2 denote the two generators of the group and τ a multiplicative substitution of order 2 with determinant -1, all ternary monomial groups may be found by combining the substitutions T_1, T_2, τ with the generators of the symmetric group of three elements. The systems of invariant forms are then found by setting up the forms which are invariant with respect to the generators of the symmetric group, taken singly or together and subjecting these forms to the substitutions T_1, T_2, τ. The conditions for invariance appear in the form of certain linear homogeneous congruences, the modulus being either the order of T_1, or of T_2. The computation of the reduced systems and the complete systems depends largely upon auxiliary quantities which appear in the solution of the congruences.

Professor Shaw exhibited a method for the development of any determinant in terms of partial determinants of the general form

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & 0 & a_{33} & a_{34} & \cdots & a_{3n} \\ a_{41} & 0 & 0 & a_{44} & \cdots & a_{4n} \\ a_{51} & 0 & 0 & 0 & \cdots & a_{5n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & 0 & 0 & 0 & & a_{nn} \end{vmatrix}$$

The central theorem of Professor Moore's paper (No. 19) on double limits, which will appear in the *Transactions*, will be sufficiently indicated by the following particular case. On the supposition that a function $f(x, y)$ is defined for

$$-1 < x < 1, \quad -1 < y < 1, \quad x \neq 0, \quad y \neq 0$$

and that the limits

$$\lim_{x=0} f(x, y), \quad \lim_{y=0} f(x, y),$$

exist and are respectively denoted by

$$h(y); \quad g(x)$$

for respectively

$$-1 < y < 1, \quad y \neq 0; \quad -1 < x < 1, \quad x \neq 0,$$

the necessary and sufficient condition that the limits

$$\lim_{y=0} h(y), \quad \lim_{x=0} g(x)$$

exist and are equal is that $f(x. y)$ approaches its limit $h(y)$ on the y-set $-1 < y < 1$, $y \neq 0$, subuniformly near $y = 0$, that is, that for every positive ε there exists a positive δ_ε ($\delta_\varepsilon \leqq 1$) and for every x, ($x \neq 0$, $|x| < \delta_\varepsilon$) there is a positive $\delta_{\varepsilon x}'$ ($\delta_{\varepsilon x}' \leqq 1$) such that

$$|f(x, y) - h(y)| < \varepsilon$$

for every (x, y) satisfying the conditions

$$|x| < \delta_\varepsilon, \quad |y| < \delta_{\varepsilon x}', \quad x \neq 0, \quad y \neq 0.$$

An equivalent condition is that $f(x, y)$ approaches its limit $g(x)$ in an analogous way.

The definition and the exposition of the elements of the theory of improper definite integrals as given by Harnack in volumes 21 and 24 (1883, 1884) of the *Mathematische Annalen*, leave much to be desired. In recent memoirs in the *Wiener Sitzungsberichte* (1898, 1899), and in the appendix to the third volume of his treatise on the calculus (1899) Stolz has given a new development of the theory, in which the absolutely convergent integrals play the leading rôle. Professor Moore's paper (No. 20), written in the spirit of the original Harnack memoirs, contains a development of the theory throwing additional light on the varying properties of the absolutely and the conditionally convergent integrals. This paper will be published in the *Transactions*.

Professor Dickson's second paper (No. 21) is also intended for publication in the *Transactions*. A set of canonical forms for the substitutions of a given group should possess the property that two of its substitutions are conjugate within the group, if, and only if, they are reducible to the same canonical form according to a definite scheme of reduction. For linear groups in a Galois field of order p^n, reduction to a canonical form belonging to a field of order p^{nr}, $r > 1$, must be possible by the introduction of new indices conjugate with respect to the initial $GF[p^n]$. The memoir gives a set of canonical forms belonging to the abelian group itself and deduces a set of ultimate canonical forms, the former depending upon the coefficients of the characteristic equation, the latter upon its roots. By these results tables are derived which give a classification into conjugate sets of all quaternary abelian substitutions and also of the operators of a single quotient group. For $p^n = 3$, the latter is the group of order 25920 occurring in the problem of the 27 lines on a general cubic surface. Its operators fall into 20 sets of conjugates: the identity, one set of period 5, 4 sets of period 3, 2 sets of each of the periods 2, 4, 9 and 12, and 6 sets of period 6.

Miss Schottenfels presented a proof of the existence of a substitution group of degree 21 and order 20160, having cyclical subgroups identical with those enumerated by Professor Dickson in the *American Journal of Mathematics*, vol. 22, p. 252, in the ternary linear fractional group of order 20160 in the Galois field $[2^2]$.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILL.

INDIRECT CIRCULAR TRANSFORMATIONS AND MIXED GROUPS.

BY PROFESSOR H. B. NEWSON.

(Read before the American Mathematical Society, February 23, 1901.)

1. In a paper* entitled "Continuous groups of circular transformations," the writer gave a list of the continuous groups of circular transformations in the plane and their chief properties. That paper treated only of direct transformations. The present paper is supplementary to the former, and deals with indirect circular transformations and the mixed groups obtained by combining these with the direct transformations.

I. Properties of Indirect Circular Transformations.

2. Certain fundamental properties of indirect circular transformations were developed by Fricke and published† in 1890. I have seen no paper of more recent date dealing with the subject. Let T be the symbol of a direct, and $\overline{T}$ the symbol of an indirect, circular transformation. Fricke's results, which form the starting point of this paper, may be stated as follows:

The second power of $\overline{T}$ is a direct circular transformation, which is either hyperbolic, parabolic or elliptic. $\overline{T}$ leaves invariant two real points, one real point, or no real points, according as its second power is hyperbolic, parabolic, or elliptic. There are three varieties of indirect transformations, viz., the hyperbolic, parabolic, and elliptic, designated by $\overline{hT}$, $\overline{pT}$ and $\overline{eT}$, respectively, distinguished according to the character of their second powers and hence also according to the number of their invariant points. $\overline{hT}$ and $\overline{hT}^2$ leave invariant the same pair of invariant points A and A'; $\overline{pT}$ and $\overline{pT}^2$ have the same invariant point A; $\overline{eT}$ has no invariant point, but interchanges the pair of points which $\overline{eT}^2$ leaves invariant. An indirect transformation of period two is an inversion of the plane with respect to a real or pure imaginary circle. Other properties of $\overline{T}$ will now be developed.

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* Bulletin (2), vol. 4, pp. 107–131 (Dec., 1897).

† Klein-Fricke's Modulfunctionen, vol. 1, pp. 196–207.

3. Since $\overline{hT}$ transforms circles into circles and leaves invariant a pair of points A, A', it must transform into itself the hyperbolic system of circles through A and A'; since hT^2 leaves invariant every circle of the system through A and A', one effect of $\overline{hT}$ must be to interchange in pairs the circles of this hyperbolic system. Such an interchange can be effected only by an inversion of the plane with respect to $\overline{C}$, one of the circles of the system. From these conditions it follows that an indirect transformation $\overline{hT}$ must be the resultant of a direct transformation hT and an inversion I on one of the path curves of hT.

The inversion I leaves invariant every point on $\overline{C}$ and every circle orthogonal to $\overline{C}$; it also produces an involutoric transformation, $k = -1$, in every circle orthogonal to $\overline{C}$. The hyperbolic system of circles through A and A' contains one and only one circle C orthogonal to $\overline{C}$. Since $\overline{C}$ and C remain invariant under both hT and I, they must also be invariant under their resultant $\overline{hT}$. Hence, *an indirect hyperbolic transformation $\overline{hT}$ leaves invariant a pair of points (AA') and a pair of orthogonal circles intersecting in A and A'.*

If k be the cross ratio of the direct transformation hT, then hT produces along the circles $\overline{C}$ and C one dimensional hyperbolic transformations characterized by k and $-k$ respectively. Two circles through A and A' making equal angles with $\overline{C}$ are interchanged by $\overline{hT}$. Thus the second power of hT leaves invariant all circles through A and A' and is therefore a direct transformation of the hyperbolic kind.

There are evidently ∞^1 indirect transformations $\overline{hT}$, one for each real value of k, each of which has the same fundamental invariant figure. When $k = 1$, $\overline{hT}$ reduces to the inversion with respect to $\overline{C}$; when $k = -1$, it becomes the inversion with respect to C.

4. By a course of reasoning similar to the above we infer the following properties of an indirect parabolic transformation: The effect of an indirect parabolic transformation $\overline{pT}$ on the points of the plane is equivalent to a direct parabolic transformation with constant a, whereby all points in the plane are moved along their path curves, followed by an inversion with respect to $\overline{C}$, one of these path curves. Hence $\overline{pT}$ leaves invariant a single point A and one circle $\overline{C}$ through A. Along $\overline{C}$ it produces a one dimensional para-

bolic transformation whose constant is a. Two circles touching $\overline{C}$ at A and making equal angles with $\overline{C}$ are interchanged. There is evidently a different transformation on the same invariant figure for each real value of a. The transformation of this system for which $a = 0$ is the inversion of the plane with respect to $\overline{C}$. Thus, *an indirect parabolic transformation $p\overline{T}$ leaves invariant a single point A and a circle through A and produces along the circle a one dimensional parabolic transformation.*

5. Similarly we see that an indirect elliptic transformation $e\overline{T}$ is the resultant of a direct elliptic transformation eT and an inversion on one of its path curves. eT leaves invariant two points A and A' and moves all points in the plane along circular path curves around A and A'. Such a transformation followed by an inversion on $\overline{C}$, one of these path curves, results in $e\overline{T}$. A and A' being inverse points with respect to every path curve of the system are interchanged by $e\overline{T}$; so that $\overline{C}$ is the only invariant figure of $e\overline{T}$. The path curves within $\overline{C}$ are interchanged with those without $\overline{C}$ in pairs. eT produces along $\overline{C}$ an elliptic transformation with cross ratio $e^{i\theta}$. There are ∞^1 indirect elliptic transformations leaving $\overline{C}$ invariant and interchanging A and A', one for each value of θ. The transformation corresponding to $\theta = 0$ is the inversion on $\overline{C}$.

The circle $\overline{C}$ is a path curve of eT and hence may be either real or a pure imaginary circle, *i. e.*, without real points. A real inversion may take place about an imaginary circle as well as about a real circle. In the hyperbolic and parabolic cases inversion about an imaginary path curve does not occur for the reason that every path curve has on it at least one real point. Thus we see that *an indirect elliptic transformation $e\overline{T}$ leaves invariant a single circle and produces along that circle a one dimensional elliptic transformation.*

Our discussion may be summed up in the following theorem :

Theorem 1. Every indirect circular transformation of the plane is the resultant of a direct non-loxodromic transformation and an inversion on one of its path curves.

II. Mixed Groups of Circular Transformations.

6. The Mixed Group mG_6. The indirect circular transformations whose properties have just been developed do

not form continuous groups, but taken with direct transformation form mixed groups. There are ∞^6 direct and ∞^6 indirect circular transformations of the plane and the aggregate of all these forms the mixed group mG_6. This general group contains many subgroups which are also mixed groups; these we now proceed to examine.

7. THE MIXED GROUP $mG_2(AA')$. We seek the aggregate of all transformations both direct and indirect which leave two points A and A' separately invariant, and also those which interchange them. Two points A and A' remain invariant under ∞^2 direct transformations which form the continuous group $G_2(AA')$. Let P and Q be two points harmonic to A and A' (whence $PQAA'$ are four points on a circle). The involutoric transformation in the group $G_2(PQ)$ interchanges A and A'. Since ∞^2 pairs of points may be chosen harmonic to A and A', we see that there are ∞^2 direct involutoric transformations interchanging A and A'.

The indirect transformations which leave A and A' separately invariant are all of the hyperbolic variety. There are ∞^1 circles through A and A'; each of these with the invariant points A and A' is the invariant figure of ∞^1 indirect hyperbolic transformations. Thus we see that there are ∞^2 indirect hyperbolic transformations leaving A and A' separately invariant.

The indirect transformations which interchange A and A' are all of the elliptic variety. There are ∞^1 circles in the elliptic system of which A and A' are the vanishing points. Each of these circles is invariant under ∞^1 indirect elliptic transformations which interchange A and A'. Thus we see there are ∞^2 indirect elliptic transformations interchanging A and A'.

The aggregate of all direct and indirect transformations which leave a point pair invariant forms a mixed group $mG_2(AA')$. From the above discussion we see that it contains two kinds of direct and two kinds of indirect transformations. The decomposition of $mG_2(AA')$ into subgroups will be discussed in articles 10 and 11.

Theorem 2. *The mixed group $mG_2(AA')$ contains ∞^2 direct transformations (which form the continuous group $G_2(AA')$), ∞^2 direct involutoric transformations, ∞^2 indirect hyperbolic, and ∞^2 indirect elliptic transformations.*

8. THE MIXED GROUP $mpG_2(A)$. We seek next the aggregate of all parabolic transformations both direct and indirect which leave a single point A invariant. We know that A remains invariant under ∞^2 direct parabolic transformations which form the continuous group $G_2(A)$. The

indirect transformations which leave a single point A invariant are all of the parabolic type. There are ∞^2 circles through A; each of these circles with the invariant point A is the invariant figure of ∞^1 indirect parabolic transformations. Hence there are ∞^3 indirect parabolic transformations leaving the point A invariant. The aggregate of all parabolic transformations both direct and indirect which leave A invariant forms a mixed group, $mpG_3(A)$. Its subgroups will be investigated in article 12.

Theorem 3. *The mixed group* $mpG_3(A)$ *contains* ∞^3 *indirect and only* ∞^3 *direct parabolic transformations.*

9. THE MIXED GROUP $mG_4(A)$. We know that there are ∞^4 direct transformations leaving a single point invariant and these form a continuous group $G_4(A)$. There are also ∞^4 indirect hyperbolic transformations having one invariant point at A; the second invariant point A' is in turn every point in the plane. There are also ∞^3 indirect parabolic transformations leaving A invariant. The aggregate of all these transformations both direct and indirect forms a mixed group $mG_4(A)$. This mixed group contains no indirect elliptic transformations.

Theorem 4. *The mixed group* $mG_4(A)$ *contains* ∞^4 *direct,* ∞^4 *indirect hyperbolic, and* ∞^3 *indirect parabolic transformations.*

10. THE MIXED GROUP $mhG_1(AA'\bar{C})$. The continuous group $G_2(AA')$ contains one continuous subgroup of hyperbolic transformations $hG_1(AA')$, whose invariant path curves are the circles of the hyperbolic system determined by A and A'. Each of these circles passing through A and A' is invariant under ∞^1 indirect hyperbolic transformations. Let C be one of these circles; then $\bar{C}$ is invariant under ∞^1 direct and also ∞^1 indirect hyperbolic transformations. These two systems form the mixed group $mhG_1(AA'\bar{C})$.

Within the mixed group $mG_2(AA')$ there are ∞^1 such subgroups, one for each circle of the hyperbolic system determined by A and A'. All of these mixed groups contain the same continuous group of direct hyperbolic transformations $hG_1(AA')$.

Theorem 5. *The mixed group* $mG_2(AA')$ *contains* ∞^1 *mixed subgroups* $mhG_1(AA'\bar{C})$, *all of which contain the same continuous group* $hG_1(AA')$.

11. THE MIXED GROUP $meG_1(AA'\overline{C})$. The continuous group $G_2(AA')$ contains one continuous subgroup of elliptic transformations $eG_1(AA')$ whose invariant path curves are the ∞^1 circles having A and A' for a pair of inverse points.

Each circle of this elliptic system is invariant under ∞^1 indirect elliptic transformations. Let $\overline{C}$ be one of these circles, then $\overline{C}$ is invariant under ∞^1 direct and also ∞^1 indirect elliptic transformations. These two systems form the mixed group $meG_1(AA'\overline{C})$.

Within the mixed group $mG_2(AA')$ there are ∞^1 such subgroups, one for each circle of the elliptic system determined by A and A'. Each of these mixed groups contains the same continuous group of elliptic transformations, $eG_1(AA')$.

Theorem 6. *The mixed group* $mG_2(AA')$ *contains* ∞^1 *mixed subgroups* $meG_1(AA'\overline{C})$, *all of which contain the same continuous group* $eG_1(AA')$.

12. THE MIXED GROUPS $mpG_2(A)$ AND $mpG_1(A\overline{C})$. The continuous group $pG_2(A)$ contains ∞^1 continuous groups $pG_1(A)$. The path curves of one of these subgroups form a parabolic system of circles through A. Each circle $\overline{C}$ of this parabolic system is invariant under ∞^1 indirect parabolic transformations; consequently the circle $\overline{C}$ and the point A on it are invariant under ∞^1 direct and also ∞^1 indirect parabolic transformations, which therefore form a mixed group $mpG_1(A\overline{C})$. The group $mpG_2(A)$ contains ∞^2 such groups $mpG_1(A\overline{C})$, one for each circle through A.

The aggregate of all the groups $mpG_1(A\overline{C})$ whose invariant circles form a parabolic system of circles through A constitute a group $mpG_2(A)$. For this parabolic system of circles remains invariant under all of these ∞^2 transformations; the system as a whole remains invariant though the individual circles of the system are not necessarily invariant under all the transformations of the group. The ∞^1 subgroups $mpG_1(A\overline{C})$ of $mpG_2(A)$ all contain the same continuous subgroups $pG_1(A)$.

Theorem 7. *The mixed group* $mpG_3(A)$ *contains* ∞^1 *mixed subgroups* $mpG_2(A)$ *and each of these contains* ∞^1 *mixed subgroups* $mpG_1(A\overline{C})$. *All subgroups* $mpG_1(A\overline{C})$ *of* $mpG_2(A)$ *contain the same continuous subgroup* $pG_1(A)$.

13. THE MIXED GROUP $mG_3(\overline{C})$. Let $\overline{C}$ be any real circle in the plane; we know that $\overline{C}$ is invariant under ∞^3 direct transformations which form a continuous group $G_3(\overline{C})$. Every direct transformation T in this continuous group followed by an inversion on the circle $\overline{C}$ results in an indirect transformation $\overline{T}$, and the aggregate of all these transformations both direct and indirect forms a mixed group $mG_3(\overline{C})$.

Every pair of points on $\overline{C}$ are the invariant points of a mixed group of hyperbolic transformations $mhG_1(AA'\overline{C})$, leaving $\overline{C}$ invariant; hence $mG_3(\overline{C})$ contains ∞^2 such mixed groups of hyperbolic transformations. Every point on $\overline{C}$ is the invariant point of a mixed group of parabolic transformations $mpG_2(A\overline{C})$, leaving $\overline{C}$ invariant; hence $mG_3(\overline{C})$ contains ∞^1 such mixed subgroups of parabolic transformations. Every pair of points inverse with respect to $\overline{C}$ are interchanged or remain invariant under the transformations of a mixed elliptic group $meG_1(AA'\overline{C})$; hence $mG_3(\overline{C})$ contains ∞^2 such mixed subgroups of elliptic transformations.

Theorem 8. *The mixed group $mG_3(\overline{C})$ contains ∞^2 mixed subgroups $mhG_1(AA'\overline{C})$, ∞^2 mixed subgroups $meG_1(AA'\overline{C})$, and ∞^1 mixed subgroups $mpG_1(A\overline{C})$.*

14. THE MIXED GROUP $mhG_2(A\overline{C})$. The mixed group $mG_3(\overline{C})$ contains ∞^2 hyperbolic subgroups $mhG_1(AA'\overline{C})$. Let the point A remain fixed and let the point A' be in turn every point on $\overline{C}$. The aggregate of all the subgroups $mhG_1(AA'\overline{C})$ whose invariant points satisfy this condition forms a mixed group $mhG_2(A\overline{C})$. This group contains one parabolic subgroup $mpG_1(A\overline{C})$, but no elliptic subgroups. The group $mG_3(\overline{C})$ contains ∞^1 such subgroups $mpG_2(A\overline{C})$, one for each point on the circle $\overline{C}$.

Theorem 9. *The group $mG_3(\overline{C})$ containing ∞^1 subgroups $mhG_2(A\overline{C})$, each of which contains hyperbolic and parabolic transformations but no elliptic transformations.*

15. THE MIXED GROUP $mG_3(i\overline{C})$. Let $i\overline{C}$ be any imaginary circle in the plane. We know that $i\overline{C}$ is invariant under ∞^3 direct elliptic transformations which form the continuous group $G_3(iC)$. Every transformation in this group followed by an inversion on the circle $i\overline{C}$ results in an indirect elliptic transformation $e\overline{T}$ and the aggregate of all these transformations both direct and indirect forms the mixed group $mG_3(i\overline{C})$. This group evidently contains ∞^2 mixed subgroups $meG_1(AA'\overline{C})$, one for each pair of points in the plane inverse with respect to the circle $i\overline{C}$.

Theorem 10. *The group $mG_3(i\overline{C})$ contains only elliptic transformations; these fall into ∞^2 subgroups $meG_1(AA'\overline{C})$.*

Résumé.—We have thus found a list of eleven mixed g roups

of circular transformations of the plane. These with their appropriate symbols and invariant figures are as follows:

1. mG_0. No invariant figure.
2. $mG_1(A)$. A single invariant point.
3. $mpG_1(A)$. A single invariant point.
4. $mG_2(\bar{C})$. A real invariant circle.
5. $mG_2(i\bar{C})$. An imaginary invariant circle.
6. $mpG_2(A)$. A single invariant point.
7. $mG_2(AA')$. An invariant point pair.
8. $mhG_2(A\bar{C})$. An invariant point and a circle through it.
9. $mhG_1(AA'\bar{C})$. A pair of invariant points and a pair of orthogonal circles.
10. $mpG_1(A\bar{C})$. An invariant point and an invariant circle.
11. $meG_1(AA'C)$. An invariant point pair and an invariant circle.

LAWRENCE KANSAS,
January 18, 1901.

PURE MATHEMATICS FOR ENGINEERING STUDENTS.

BY PROFESSOR A. S. HATHAWAY.

(Read before the Chicago Section of the American Mathematical Society, December 28, 1900.)

I. ITS UTILITY.

I HAVE had opportunity to become acquainted with the written opinions of graduates of from one to many years' standing, with regard to the benefit and utility of their instruction in all departments of technical work, and have also taken the opportunity of conferring personally with graduates in respect to my own department. In discussing this subject, I shall therefore appear, not as a special pleader for pure mathematics, but as one who proposes to present its claims in their true proportions to the other necessary work of the student.

In the first place, mathematical analysis is not so directly useful to the average engineer as the mathematician might expect. The mathematics that an engineer is obliged to use regularly is of that cut and dried form which is found tabulated in engineering handbooks of easy access, so that only a little arithmetic or algebra, and occasionally some

trigonometry, answer all practical purposes. Seventy-five per cent. of graduates of technical institutes, now in actual practice, have probably forgotten how to differentiate and integrate, although most of them could easily refresh their memory in this respect if it were necessary. We can say, in favor of the calculus, that the remaining twenty-five per cent. include those who are leaders in their profession, and those who occupy positions under which new conditions are constantly arising and in which originality of thought is required; and among these are many who have found use for all the mathematics that they could acquire.

The question arises, when so few engineering students afterwards make practical use of the higher mathematics, Is it advisable to teach it thoroughly to all students? The answer that I find to this question is, yes. Almost every practical engineer who has been properly taught, sees the benefit of such studies. Such engineers will generally grant of their own accord the practical benefits of their mathematical training, even though they have had no occasion to use the analysis as an instrument of investigation. Many wish that they could use it better, as they see opportunities where it might be of advantage, if it were at their fingers' ends; and all agree that, aside from its practical use, there is a training in habits of thought, points of view, and intellectual comprehension of ordinary engineering problems, which only the study of the higher mathematics can give. The fact is that the fundamental ideas of natural phenomena that appear in all technical work are found in their most logical and clearest form in the higher mathematical analysis. The development of the mind in logical relations whose analogues exist in some form or other in nature, creates that mental insight which perceives the dominant ideas and develops the most practical methods of treatment of the problems of each day.

The technical training that is involved in the study of the higher mathematics, particularly the calculus, therefore answers the question as to its value and utility for all engineering students, irrespective of any direct value of the analysis as an instrument of investigation.

II. Methods of Instruction.

Some practical men who look upon mathematical analysis only as an instrument of the engineer, would have it taught solely for the purpose of obtaining the requisite mechanical skill in its use. It is to be made a matter of memorizing

formulas and methods, while any explanation is recommended that placates the mental comprehension of the student, whether it is logical and common sense or not, so long as it is brief. This is not only depriving the majority of students of the real benefits that accrue to them from the study of mathematics, such as in the acquirement of fundamental ideas of technical value, but it is also a dangerous method of instruction. When a young engineer can do something supposed to be of value in his profession, but about which he has only certain vague conceptions and no perfect understanding from a common sense point of view, it is quite natural that he should give it an exaggerated importance; and by an equally obscure mental process, he obtains an exalted opinion of his own capabilities. The result is frequently disastrous to himself in some important work that he may have been led to undertake in the fancy that he was equal to it. The frequency of such cases, as coming from the graduates of a certain school, is certain evidence of faulty methods of instruction, particularly in mathematics. One difference between rigorous and logical methods of instruction, and memory training, is in their effect on what is called the personal equation of the student, *i. e.*, his liability to make mistakes. The first methods bring such equation before the student, show that he is in the habit of superficial thinking and of readily accepting plausible conclusions that are more often wrong than right. He is forced to develop a system of checks upon his ideas and conclusions, and to exercise a continual care in discovering and eradicating the effects of those lapses of mental acumen to which all are more or less liable according to the training of the individual and the temporary condition of the mind. On the other hand, training by memory alone leaves the individual in ignorance of his unconscious errors of thought, and while he may be very expert in his lessons, yet in the elements of detailed and accurate work, he may prove nothing but a blunderer, with all actual mental training to begin over again in practical life if he is to succeed.

Mathematical analysis should therefore be taught according to the precepts of good logic and common sense. It is not necessary to consider all the detailed refinement of the pure mathematician, but some of the consequences of that refinement should not be neglected, and above all the student should learn that some very good methods of demonstration for ordinary cases do have their exceptions. A combination of the recitation system, based upon a good text book, with the lecture system seems best, both because

it secures that flexibility of course hereafter considered, and because a proper amount of lecture work is in itself a training in quickness of apprehension on the part of the student, and creates the ability to seize upon the important parts of a subject. In fact, extensive note-taking of lectures on the part of the student should not be encouraged, the idea being to work up the subject afterwards from as few notes as possible.

The working of examples, especially numerical ones, cannot be too excessive, if within the time of the student. In this connection, most students can, by proper thought given to methods of study and work, reduce the time necessary to be spent on that subject by fully one-half the usual amount spent unmethodically. This means, perhaps more time spent when beginning a subject, in learning thoroughly its fundamental principles and methods, and less time later on—just the reverse, it will be seen, of the usual procedure of the average student.

III. The Course.

With reference to the framework of a technical mathematical course, I should leave it as it is in most institutes —algebra, geometry, trigonometry, analytical geometry, the calculus, and the elements of differential equations with particular reference to those equations occurring in engineering subjects. Under geometry should be included the elements of projective geometry, especially of parallel projection, with applications to the conic sections. The calculus cannot be begun too early, before analytic geometry even. The calculus is the fundamental topic; its ideas are so useful in mental development that when they are thoroughly mastered, other subjects are more easily acquired, so that it is in reality a time-saver. With us also, the last term of mathematical instruction is given up to the student's own devising in the way of subjects and development. Each student is expected to prepare what is essentially a course of lectures on chosen topics before the class. His aids in this work are whatever suggestions and information he can acquire from sources within his reach—the library, text books, his teachers. The result has been an improvement on knowledge already gained, and a development of pure and applied mathematics on the part of the student far beyond the usual course. Remembering that each student selects his own line of work, and hears, and is naturally interested in, whatever his classmates present, I

have come to consider this term's work as the most valuable of all.

A course should be made flexible, by introducing at suitable times and places subjects that are interesting and likely to prove valuable in practical work. All pure mathematics is valuable as mental discipline; but the adaptability of different kinds will change with the years, the class, the character of the students, and the particular interests that they may be led by circumstances to manifest. For instance, I had, from a recent junior class, a request for a course in determinants, which is only briefly considered under algebra. The reason for this request was that the professor in dynamical engineering was at the time considering the subject of transformer problems, such as the problem of distribution of alternating current over a long line having capacity with up and down transformers at either end, and had found the determinant notation extremely useful in this work.

As another illustration of the changes which time brings, take the case of the applications of the imaginary analysis. The continuous current has had its day, although the practical applications of Faraday's discovery in 1832 was delayed nearly half a century by the want of a practical method of converting the alternating current into a continuous one. Recent discoveries have made the alternating current the more valuable in practical applications; and in this current we find a beautiful analogue of the imaginary analysis—so much so that technical writers are using that analysis in their publications on alternating currents for practical engineers. The imaginary analysis should therefore be taught to-day in live technical schools, with special reference to its geometrical and physical interpretations.

Instances might be multiplied showing that new circumstances will continually arise that necessitate deviations from old and established lines of instruction. Such flexibility in the course can only be secured, without continual change in its framework, through the lecture system in connection with text book instruction.

IV. The Instructor.

The success of a course, in mental training, and in meeting the practical needs of the engineer, depends very largely upon the instructor. He must be one who keeps up an interest in pure mathematics and their practical applications, and who is also clear in his own ideas, so that he may know

what is interesting and useful, and be able to introduce it with clearness and brevity when occasion offers. He should, in other words, be a high grade man—not the highest, for that is impossible for all schools to secure, but certainly of such reputation among pure as well as applied mathematicians as causes him to be regarded with respect by the leaders in those subjects.

There is an objection in certain technical schools to a mathematical teacher who has too much fondness for pure mathematics. I once heard the president of such a school not only express this objection, but go further, by saying that he did not want a man in his faculty who knew more than himself on any subject.

The principal cause of the above objection to higher mathematical knowledge seems to be a fear that it leads to talking over the heads of the students. I will contend that it is an advantage to a school to have instructors who are able to talk over the heads of their students, and who do it occasionally. When an instructor talks over his head, the student may have some very strong feelings as to the futility of the exhibition, but he will find that the ideas do not always fly over. Like seeds planted in soil they await the germinating time. There is an unconscious impression made upon the mind by listening to ideas that are from a higher point of view than our own, and such impressions are often more durable than those left by apprehended impressions, and finally, as the result of gradual ascent to the higher point of view, they become fully understood and appreciated.

It is of course not possible to rely upon the results of such influences for the instruction of a technical student, and some ability to make himself understood and bring his mind within the confined horizon of the student, is necessary in the occupant of the instructor's chair. Such an instructor who also attempts to lift the students above their too small horizon at the certain risk of talking sometimes over their heads, produces, for some reason, students of higher acquisition, and of greater enthusiasm and practical ability, than is possible for the drill master to accomplish with his rule of thumb. It takes time to determine the influence of a teacher, and in the end the conscientious student is bound to recognize the superior influence upon his mental development of the instructor of superior attainments.

ROSE POLYTECHNIC INSTITUTE.

ADAMS'S UNPUBLISHED PAPERS.

The Scientific Papers of John Couch Adams, Vol. II. Part I., *Extracts from Unpublished Manuscripts*, edited by R. A. SAMPSON. Part II., *Terrestrial Magnetism*, edited by W. G. ADAMS. Cambridge, Pitt Press, 1900. 4to. xxxii + 646 pp., with 6 magnetic charts.

Lectures on the Lunar Theory. By JOHN COUCH ADAMS, edited by R. A. SAMPSON. Cambridge, Pitt Press, 1900. 8vo. 88 pp.

BY those who knew John Couch Adams best the examination of the manuscripts which he left and the publication of a selection from them has been looked forward to with much interest. His well known reluctance to publish anything not in a complete form—"I have some finishing touches to put to it"—raised hopes that valuable results might be contained in the packets of neatly-written and dated papers which were turned over after his death for examination. It was known that he had been at work on various problems, in particular on that most difficult one—the theory of Jupiter's satellites, and on the theory of terrestrial magnetism. It was known, too, that Adams rarely attacked a problem without throwing a new light upon it or evolving some unexpected result. He had not fallen into the modern habit of making "preliminary communications," or, at least, if these were made they were verbal and did not find their way into print. Thus the publication of this volume was looked forward to with greater expectation than is usual in the case of other scientists who have lived to his age.

I shall not be misunderstood in saying that a slight feeling of disappointment arises on turning over its pages and seeing the matter contained therein. Adams's reputation as a mathematician and astronomer of the highest class rests on too firm a basis to be disturbed by anything which may or may not be contained in his unpublished papers. His published work, small in quantity though it may be, made an ineffaceable mark on the subjects which he touched. The discovery of Neptune, the determination of the accurate value of the secular acceleration of the moon's mean motion, the method by which he arrived at the correct period of the November meteors are sufficient to stamp the character of his work without mentioning other results which show equal

ability but are not so well known. If we are disappointed at finding no great discovery in the new volume or anything indicating that he was on the track of one, we have no reason to regret its appearance or to feel that the labor and thought which the editors have expended in putting the manuscripts into form have been in any sense wasted. There is, in fact, almost nothing which is not new. Some of the papers consist of new methods in handling old problems, others consist of the detailed work in the solution of problems, the mere results of which had alone been published, and, more important than all these, his great work on terrestrial magnetism is set forth in a complete and connected form.

The first number consists of a selection from Adams's lectures on the lunar theory. These have also been published separately without change for the use of those to whom the larger volume might not be easily accessible. The editor, Professor R. A. Sampson, has not aimed at giving the lectures in the complete form in which Adams delivered them. Portions which contained alternative proofs or explanations which are to be found in the text books have been omitted. The result is a continuous development which gives the coefficients of the principal inequalities due to the sun's action with a good degree of approximation. In his preface, Professor Sampson says "Of current elementary theories it may be said that they leave off where the difficulties of the subject begin, that is to say, where the various cases of slow convergence have been exposed, but not dealt with. It is perhaps not too much to say that these lectures carry us to the point where such difficulties end, in an adequate evaluation of all the chief constants. They leave the problem effectively solved and not merely stated, and show the path clear for the formation of a detailed theory, if that is desired." This is hardly correct as to the question of convergence, or at any rate it obscures the issue. The difficulty of slow convergence can hardly be said to have been dealt with by Adams any more than by his predecessors. That he has obtained close approximations is due to the fact that the numerical value of m,* the ratio of the mean motions, is used instead of series proceeding in powers of m. There is not much difficulty about convergence in the case of the moon when the numerical value of m is used at the outset, since slow convergence only occurs along powers of m. Again, one of the chief difficulties of the

* More accurately, he uses $m/(1-m)$.

subject is the calculation of the higher terms with sufficient accuracy. Owing to small divisors the earlier terms have to be calculated with far greater accuracy than would be necessary if these small divisors did not arise, and this difficulty has not been bridged in any other way than by calculating the earlier terms to a high degree of accuracy.

The method used by Adams is practically the same as that of G. W. Hill, except that he has adopted polar where Hill uses rectangular coördinates. For the earlier inequalities this method works well enough, but it would most probably give rise to unmanageable calculations for the higher ones if a complete theory were in contemplation. In general, it lacks the elegance and capacity for high approximation with a given amount of labor which characterizes Hill's methods, with one or two exceptions, notably that by which Adams arrives at the equation from which the motion of the perigee is obtained. In fact, the symmetry possible with rectangular coördinates disappears altogether when polar coördinates are used, and in the latter case the expansions of such expressions as

$$\cos (a_0 + a_1 \cos t + a_2 \cos 2t + \cdots)$$

have to be continually performed while, in the former, we have only to do with multiplications of algebraic series.

The next paper contains the detailed work necessary for the expression of the infinite determinant giving the motions of the node or perigee. In No. 3, entitled "Numerical developments in the lunar theory," we have the numerical values of the coefficients of δn when we put $n + \delta n$ for n and $n' + \delta n'$ for n' with $\delta n = \delta n'$ in the variational inequalities. The values of the coefficients of the parallactic inequalities are also obtained to 16 places of decimals, the first power of the ratio of the parallaxes only being retained. The determination of the part of the secular acceleration of the moon's mean motion which depends on m only is found in the next paper, and in No. 15 without having recourse to expansions in powers of m. Adams's method is in reality a numerical one and depends on the changes in the values of the coefficients of several periodic terms, the principal ones being those found in the previous paper. The writer has shown how this part of the secular acceleration can be found from the *constant* term in the parallax with any degree of accuracy required.*

Proc. Lond. Math. Soc., 1896.

From these papers it appears that Adams had calculated the following sets of inequalities with the accuracy necessary for a complete lunar theory: The parts of the variational terms which depend on m only and those which depend on m and the second power of the solar eccentricity; the terms depending on m and the first powers of the solar eccentricity, the ratio of the parallaxes, and the inclination; and the principal part of the motion of the node. Other inequalities, *e. g.*, the terms depending on m and the first power of the lunar eccentricity had been found to a less degree of accuracy; some of these appear in the lectures.

A short note on Neison's lunar inequality due to Jupiter follows. Adams finds the theoretical coefficient of this with some accuracy by considering it as a sort of Jovian evection, thus making use of the literal expression for the solar evection. The value obtained differs by about 15 per cent. from that of M. Radau,* who has calculated a large number of the planetary inequalities.

No. 6 contains a novel suggestion for the solution of the equation

$$\frac{d^2w}{dt^2} + (a_0 + a_1 \cos 2t + a_2 \cos 4t + \cdots)w = 0$$

which occurs so frequently in celestial mechanics. The solution is of the form

$$w = \Sigma A_i \cos \{(k + 2i)t + \alpha\},$$

and the difficulty of calculation arises from the presence of small divisors due to the fact that k is in general very nearly equal to unity. Adams's method really amounts to finding all the other coefficients in terms of k, A_{-1} in such a way that no small divisors occur; the two equations for these quantities are solved as a final step.

Papers 7, 8, 9, 12 are on Jupiter's satellites. As Professor Sampson has presented Adams's investigations, it appears that a new method of treating the whole subject was in contemplation, somewhat analogous to that which he used for the motion of the moon. He first solves the problem of the motion of a satellite about an oblate primary, in fact he finds the inequalities due to the figure of Jupiter. The equations for the three coördinates are reduced to two simultaneous ones of the second order with the distance and perpendicular on the equatorial plane as dependent vari-

* *Ann. de l' Observ. de Paris*, vol. 21.

ables and the time as independent variable. The equations are solved literally so that they can be applied to any satellite. The third coördinate is then easily found, as the editor shows in a short note. The development of

$$[a^2 - 2aa' \cos (nt - n't + \varepsilon - \varepsilon') + a'^2)]^{-\frac{1}{2}},$$

necessary to obtain the mutual actions of the satellites on one another, is then made with some completeness and applied, for illustration, to the first two satellites. The mutual perturbations of two satellites when the eccentricities and inclinations are neglected next follow. Here again polar coördinates are used so that the perturbations are found directly, instead of indirectly by the more usual method of the variation of arbitrary constants. Values for the radius vector and the longitude, of the proper form, are substituted in the differential equations and the coefficients are determined by equations of condition.

As Adams does not go beyond the first order of the disturbing forces in the published investigations, the difficulty arising from the exact relation

$$n - 3n' + 2n'' = 0,$$

where n, n', n'' are the mean motions of the first three satellites, does not come into consideration. In fact, as far as the theory is given, we have simply those inequalities which are analogous to the variational and parallactic terms in the lunar theory. Some numerical results are given. It would be interesting to see how this method of procedure would work out when applied to attack the complete problem. Perhaps Adams did not see his way clear for this and therefore refrained from publishing what he had already done. The chief difficulties undoubtedly arise mainly in the terms depending on the eccentricities and on the square of the disturbing forces owing chiefly to the smallness of $n - 2n'$, $n' - 2n''$ and to the exact equality of these two expressions. The other papers on this subject contain corrections of the masses of the satellites and of errors in Damoiseau's and Laplace's theories.

Every astronomer knows of H. A. Newton's investigations on the orbit of the November meteors in which he arrived at the conclusion that the observations could in general be satisfied by any one of five competing values for the mean period. Adams supplemented this by showing that only one of these five values was consistent with the observed motion of the node of the orbit. In paper 10 we have the

details of the method by which he arrived at the result. The brevity of the work is remarkable. The calculations are not given in detail. They are reduced to mechanical quadratures and a few of the principal steps for the perturbations by Jupiter are set forth. The method is a modification of Gauss's formulæ.

Some extracts from Adams's lectures on the Figure of the Earth occupy No. 11. They consist of theorems on the attractions of spheres and spheroids when the law of force is any function of the distance and conclude with the well-known differential equation connecting the ellipticity of strata with the density. A couple of notes on the analytical interpretation of Newton's methods for finding the variation and motion of the apse, a short method for obtaining certain of Laplace's expansions of functions in power series, and lists of errata in the works of Plana, Damoiseau, and de Pontécoulant occupy the remaining pages of Part I. In the last there are one or two slight errors. On p. 236, line 3, for 1025, read 1024; omit line 5, which is given on the last line but one of the previous page; on line 7 insert a + before the last number.

The second part, containing Adams's work on terrestrial magnetism, occupies two-thirds of the volume. The difficulties of Professor W. G. Adams, who undertook to edit this portion of the work left by his brother, must have been sufficiently great The matter had not been put into final form, though it was probably ready for it. The editor's chief labor was to gather together the various packets of theoretical and numerical work; to discover their connection, and by blending them to set forth in a complete manner the results of Adams's labors. A glance at the memoir shows that this task can have been no light one, and students will be grateful to Professor W. G. Adams for the manner in which he has performed it. In a preface he gives a summary which shows well the nature of the work and the conclusions reached by his brother.

The method adopted is that of Gauss, with modifications. But instead of the 24 constants with which Gauss contented himself, Adams has included 240, just ten times as many. Moreover, the observations which in Gauss's time were few and badly distributed, have increased much in volume, though still far from what is really necessary for a thorough determination. The memoir is arranged in eight sections, of which the first four consist of various relations between the harmonics and coefficients used in such investigations. Sections V, VI, respectively treat of the theory of

terrestrial magnetism when the earth is considered as a sphere and as a spheroid. The last two sections contain the numerical calculations. At the end are maps drawn from the numerical results for every 5° of latitude and every 10° of longitude. The agreement with observation is stated by the editor to be very satisfactory.

In conclusion, the reader will gather from the rough indications here given that the volume is not a mere compilation of incomplete fragments. Whatever he may think about the value of collections of published papers, and in particular of Volume I of Adams's works, it will be immediately apparent that Volume II is at least on the same footing as any other book containing original and previously unpublished scientific work, and its possession by scientific libraries will be as much a necessity as is that of the best known journals and treatises.

ERNEST W. BROWN.

NOTICE SUR M. HERMITE.

PAR M. C. JORDAN.

ADDRESS DELIVERED AT THE MEETING OF THE PARIS ACADEMY OF SCIENCES, JANUARY 21, 1901.

THE French school of mathematics loses in the person of M. Hermite its head and master.

It would be rash to undertake to analyze in haste and under the stress of keen emotion the long series of his works which have thrown so much lustre on the second half of the nineteenth century. Such an undertaking calls for more time and calmer feelings. Addressing then to our venerated confrère the last farewell, which his modesty forbade pronouncing at his grave, we limit ourselves here to pointing out, in broad lines and as far as memory permits, some of the discoveries which we owe to him.

In 1843, M. Hermite entered the Ecole Polytechnique at the age of twenty years. At the suggestion of Liouville, he wrote to Jacobi communicating the results which he had obtained relative to the division of abelian functions, then but little known. The illustrious German geometer, who was occupied at the time with the editing of his works, did not hesitate to give the letter of his young correspondent a place beside his own investigations. He wrote to him a little later: "Do not be troubled, Monsieur, if some of

your discoveries are to be found in my former researches. As you commenced where I finish, there is necessarily a small sphere of contact. In the future, if you honor me with your communications, I only shall have to learn."

The prediction of the great geometer was not long in being fulfilled. In the four letters which follow and which Jacobi has also preserved for us, M. Hermite proposed first to generalize the theory of continuous functions; but he shortly found himself led to vaster problems in the arithmetical theory of forms, where he soon obtained admirable results. At the very beginning of his works he points out several methods for reducing quadratic forms in any number of indeterminates. A little later the introduction of continuous variables leads him to the discovery of deeper lying truths.

He gives the complete solution of the problem of the arithmetical equivalence of general quadratic forms or of forms decomposable into linear factors; he determines the transformations of these forms into themselves; he demonstrates, in a manner altogether novel and purely arithmetical, the celebrated theorems of Sturm and Cauchy on the separation of the roots of algebraic equations. He introduces the fertile notion of quadratic forms with conjugate variables, and deduces from their theory a new demonstration of the beautiful theorem of Jacobi on the number of decompositions of a number into four squares. He arrives finally at the remarkable proposition that the roots of all algebraic equations having integral coefficients and the same discriminant can be expressed by a limited number of distinct irrationalities.

The algebraic study of forms is also the object of his meditations. The notion of invariant which dominates this theory had remained a little confused until the day when Cayley threw it into full light in the celebrated memoir dated 1845. Cayley, Sylvester, and Hermite divided the new domain which opened before them. Their works are so interlaced in this fraternal rivalry that it would be difficult and hardly desirable to distinguish with precision the part done by each in their common work. It seems, however, that we can attribute in particular to M. Hermite the law of reciprocity, the discovery of associated covariants, that of skew invariants, and the formation of the complete system of covariants of cubic and biquadratic forms and of invariants of the form of the fifth order.

These important researches in arithmetic and algebra did not satisfy his activity; he pursued at the same time his

studies on transcendants. In a series of memorable investigations he solved the problem of the transformation of hyperelliptic functions, and from the developments in series of the elliptic functions he deduced important formulæ relative to the number of classes of quadratic forms.

He laid the foundations of the theory of modular functions at the same time and solved in its details the difficult question of their transformation, thus giving far in advance a model for those who have elaborated and generalized this theory in our day.

The impression produced on geometers by the ensemble of these works is reflected very well in a picturesque phrase which we received at that time from the lips of Lamé: "To read the memoirs of M. Hermite makes one's flesh creep."

In 1856, at the age of thirty-four years, M. Hermite entered the Institute; in 1862 a chair was created for him at the Ecole Normale; a little later he became professor also at the Ecole Polytechnique and at the Sorbonne.

At this epoch the course of instruction, it must be said, was hardly up to date. The great discoveries by which Gauss, Abel, Jacobi, and Cauchy had transformed the science during half a century were passed over in silence, as if they were of interest only to the initiated few. M. Hermite cast them boldly into the public domain. This happy audacity has borne its fruits. Witness our young and brilliant school of geometers; they were all students of Hermite, and owe a great deal of their success to his lectures and kindly encouragement.

His peaceful realm was not confined to our borders. M. Hermite kept up correspondence throughout learned Europe, and young talent anywhere could count on his counsel and support. Neither the duties of instruction nor even the afflictions of age could do injury to the fertility of his mind. In fact, from this second period date a great number of beautiful works which yield precedence in no way to the works of his youth. However, an appreciable evolution took place in the object of his investigations. Arithmetic and algebra which had predominated up to this time gave way to the integral calculus.

The transition is made by a celebrated memoir on the equation of the fifth degree, whose solution he gives by means of elliptic functions. Then follow the researches on interpolation, on new methods of developing functions in series of polynomials, on the discontinuities of definite integrals which depend on a parameter, etc.

In the theory of elliptic functions M. Hermite discovered a fundamental formula which permits their decomposition into simple elements, and consequently their integration. He was the first to study doubly periodic functions of the second species.

We arrive finally at the memoir on the exponential function, worthily crowning his long researches on the developments in continued fractions. He made clear that the number e is transcendental. Lindemann has since established that the number π is also transcendental. The solution of the problem of the quadrature of the circle, so vainly sought throughout all the centuries, is therefore demonstrated to be impossible.

We can legitimately claim for M. Hermite a share in this beautiful result, because it has been attained by following the process which he employed for the exponential. But one would give a very incomplete idea of the rôle of great minds by measuring them exclusively by the new truths which they have explicitly enunciated. The methods which they have bequeathed to their successors, leaving to their care the application of these methods to new problems perhaps unanticipated by themselves, constitute another part, sometimes the principal part, of their glory, as the example of Leibnitz shows.

For almost a century we have labored to develop the fertile germs which Gauss and Cauchy have sown in their writings; it will be the same with Hermite. Behold two examples to prove it:

The remarkable group of substitutions which he encountered in his researches on the transformation of abelian functions serves as the essential element for the solution of a problem altogether different, the resolution of equations by radicals. It appears again in the discussion of the second variation of definite integrals.

The quadratic forms having conjugate variables are the indispensable fundament of investigations in the reduction of the most general forms with real and complex coefficients.

M. Hermite loved science for its own sake and was but little occupied with its applications; these came spontaneously and by superaddition. To Lamé's equation, whose integration constitutes the last of his great works, he attached quite a series of problems in mechanics: rotation of a solid, determination of the elastic curve, oscillations of the conical pendulum.

To form a correct idea of the place occupied by M. Hermite in the mathematical world, one should have taken

part with us in the memorable fêtes of his jubilee in 1892. All his friends, his disciples, his admirers shared in this impressive ceremony ; all the learned societies of Europe sent either addresses or delegates.

The same year saw the jubilee of Pasteur. To-day Pasteur and Hermite are no more ; there remains for us only the souvenir of their examples and their works, but these are sufficient to immortalize their memory.

Permit us in concluding to express a wish in behalf of the section of geometry. The work of Hermite is very scattered; in addition to the principal memoirs, it contains many letters and short notes disposed here and there ; but all bear la griffe du lion. The Academy would honor itself and render a great service to geometers by undertaking the publication of the complete works of Charles Hermite.

NOTES.

THE committee in charge of the colloquium to be held in August in connection with the summer meeting of the AMERICAN MATHEMATICAL SOCIETY is now able to announce that Professor OSKAR BOLZA has consented to give a course of lectures on "The calculus of variations, in particular Weierstrass's discoveries."

THE ninth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago, on Saturday, April 6, 1901, the first session opening at 10 o'clock A. M., in the Ryerson Physical Laboratory. The Christmas meeting will be held at Northwestern University, Evanston, Ill., on Friday and Saturday, December 27 and 28. Titles, abstracts, and time requirements of papers to be read at the April meeting should be in the hands of the Secretary of the Section, for the use of the programme committee, not later than March 20.

THE January number (volume 23, number 1) of the *American Journal of Mathematics* contains the following papers: "Die Typen der linearen Complexe rationaler Curven in R_n," by S. KANTOR; "Transformations of systems of linear differential equations," by E. J. WILCZYNSKI; "Distribution of the ternary linear homogeneous substitutions in a Galois field into complete sets of conjugate substitutions," by L. E. DICKSON; "Distribution of the quaternary linear homogeneous substitutions in a Galois

field into complete sets of conjugate substitutions," by T. M. PUTNAM; "On the determination and solution of the metacyclic quintic equation with rational coefficients," by J. C. GLASHAN; "Construction of the geometry of euclidean n-dimensional space by the theory of continuous groups," by E. O. LOVETT; "A table of class numbers for cubic number fields," by L. W. REID; "On certain properties of the plane cubic curve in relation to the circular points at infinity," by R. A. ROBERTS. The number also contains portraits of Professors GEORGE SALMON and MITTAG-LEFFLER.

THE following papers were read at a meeting of the London mathematical society held on January 10, 1901: "On streaming motions past cylindrical boundaries," by Professor A. E. H. LOVE; "On some cases of the solution of $x^{p^n-1} \equiv 1$, mod. p," by Professor F. S. CAREY; "On the zeros of Bessel's functions," by Mr. E. W. BARNES; "A proof of the third fundamental theorem in Lie's theory of continuous groups," by Mr. J. E. CAMPBELL.

AT a meeting of the Edinburgh mathematical society held on December 14, 1900, Dr. J. A. THIRD read a paper on triangles triply in perspective, and Professor R. E. ALLARDICE communicated a paper on four circles touching a common circle. At the meeting of the same society held on January 11, 1901, Professor ALLARDICE read a paper on the nine point conic.

SYLVESTER PRIZE. The first award of the Sylvester Prizes of the Johns Hopkins University (see the present volume of the BULLETIN, page 191) was made on February twenty-second. The prize is a handsome bronze medallion of the late Professor Sylvester, framed in oak. In the course of the ceremonies, President GILMAN announced the award as follows: "The first impression of this tablet is presented to LORD KELVIN, who lectured here on 'The nature of light' in 1884. This will be offered to Lord Kelvin, at the time of the jubilee of the University of Glasgow in June next, by a delegate from this university. The second copy of the tablet is now offered to Professor SIMON NEWCOMB, a distinguished astronomer, who has been a friend of the university from its inception, and who guided the affairs of the mathematical department for many years."

THE Prussian academy of sciences has conferred the Helmholtz medal upon Sir GEORGE GABRIEL STOKES. The

medal has hitherto been conferred only on Professor VIRCHOW and LORD KELVIN.

THE well known treatise on analysis of Professor E. PICARD is to appear in a new edition, of which the first part of the first volume has just come from the press of M. Gauthier-Villars, of Paris.

PROFESSOR GINO LORIA has issued a reprint of an academical address dealing with the history of mathematics which he recently presented to the University of Genoa, under the title "The transfigurations of a science."

IT is announced that Professor A. MAYER is to retire from the editorial board of the *Mathematische Annalen*, and that Professor D. HILBERT is to take his place. The board will thus consist of Professors F. KLEIN, W. DYCK, and D. HILBERT.

UNIVERSITY OF MUNICH.—The following courses in mathematics are offered during the summer semester, 1901: By Professor G. BAUER: Theory of algebraic plane curves; Mathematical seminar.—By Professor F. LINDEMANN: Analytical geometry of space; Definite integrals and Fourier's series; Mathematical seminar.—By Professor H. SEELIGER: celestial mechanics, second course, methods of Jacobi and Hamilton; Exercises with the instruments of the observatory.—By Professor A. PRINGSHEIM: subjects to be announced later.—By Dr. K. DÖHLEMANN: Descriptive geometry, second course, with exercises; The imaginary in geometry.—By Dr. E. VON WEBER: Encyclopædia of elementary geometry; Modern geometric theories.—By Dr. K. SCHWARZSCHILD: Methods of interpolation and mechanical integration.

PROFESSOR M. CANTOR, of Heidelberg, has been elected a correspondent of the St. Petersburg academy of sciences.

PROFESSOR H. S. WHITE, of Northwestern University, has received leave of absence and will remain abroad until October.

THE death is announced of Professor M. F. KOVALSKIJ, professor of mathematics at Charkow.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CANDY (A. L.). The elements of analytic geometry. Lincoln, Neb., published by the author, 1900. 8vo. 303 pp. Cloth. $2.00

FORMULAIRE de mathématiques, publié par G. Peano. Edition de l'an 1901 (tome III de l'édition complète). Turin, Bocca, 1901. 8vo. 8 + 231 pp.

HAYES (E.). Calculus, with applications; an introduction to the mathematical treatment of science. Boston, Allyn & Bacon, 1900. 12mo. 7 + 162 pp. Cloth. $1.20

HILBERT (D.). Les principes fondamentaux de la géométrie. Traduit par L. Laugel. Paris, Gauthier-Villars, 1900. 4to. 114 pp.

LAUGEL (L.). See HILBERT (D.).

LEJEUNE-DIRICHLET (P. G.), Die Darstellung ganz willkürlicher Funktionen durch Sinus- und Cosinus-Reihen (1837). und SEIDEL (P. L.), Note über eine Eigenschaft der Reihen welche diskontinuierliche Funktionen darstellen (1847). Herausgegeben von H. Liebmann, Leipzig, Engelmann, 1900. 12mo. 58 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 116.) Cloth. M. 1.00

LIEBMANN (H.). See LEJEUNE-DIRICHLET (P. G.).

PEANO (G.). See FORMULAIRE.

PIERPONT (J.). Galois's theory of algebraic equations. Six lectures delivered at the Buffalo Colloquium in September, 1896. (Published in the *Annals of Mathematics*, ser. 2, Vols. 1 and 2, 1900.) Cambridge, Harvard University, 1900. 4to. 67 pp. $0.75

PUZYNA (I.). Teorya funkcji analitycznych (Theory of analytic functions). Part 2. Lemberg, 1900. 8vo. 16 + 693 pp. M. 16.00

REID (L. W.). Tafel der Klassenanzahlen für kubische Zahlkörper. (Diss.) Göttingen, 1899. 8vo. 75 pp.

SAILER (E.). Die Aufgaben aus der Differential- und Integralrechnung und aus der analytischen Geometrie, welche bei der Prüfung für das Lehramt der Mathematik und Physik an den k. bayerischen humanistischen und technischen Unterrichtsanstalten in den Jahren 1873 bis 1893 gestellt wurden. München, Ackermann, 1900. 8vo. 187 pp. M. 4.80

SEIDEL (P. L.). See LEJEUNE-DIRICHLET.

TOWNSEND (J. T.). Ueber den Begriff und die Anwendung des Doppellimes. (Diss.) Göttingen, Dieterich, 1900. 8vo. 7 + 78 pp.

II. ELEMENTARY MATHEMATICS.

ALASIA (C.). Esercizi ed applicazioni di trigonometria piana, con 400 esercizi e problemi proposti. Milano, Hoepli, 1901. 8vo. 11 + 291 pp. (Manuali Hoepli.)

ARZANI (G.). Nozioni elementari di algebra, ad uso della terza classe tecnica. Moncalvo, Sacerdote, 1900. 16mo. 119 pp.

BORTOLOTTI (E.). Nozioni pratiche di geometria per le scuole complementari. Roma, soc. edit. Dante Alighieri, 1900. 8vo. 140 pp. Fr. 1.75

BOS (H.). Éléments de géométrie ; avec la collaboration d'A. Rebière. 5e édition. Paris, Hachette, 1900. 8vo. 503 pp. Fr. 7.00

CARTON. Eléments de géométrie. Edition adaptée spécialement au programme du premier examen du baccalauréat de l'enseignement secondaire classique. 5e édition. Paris, Poussielgue, 1901. 8vo. 316 pp. (Alliance des maisons d'éducation chrétienne.)

CHIAPPETTI (F.). Contributo allo studio del tronco di cono. Imola, Galeati, 1900. 8vo. 10 pp.

HOLZMÜLLER (G.). Elemente der Stereometrie. Teil 1 : Die Lehrsätze und Konstruktionen. Neue [Titel-] Ausgabe. Leipzig, Göschen, 1900. 8vo. 11 + 383 pp. M. 6.00

——. Teil 2: Die Berechnung einfach gestalteter Körper. Mit zahlreichen Uebungsbeispielen. Leipzig, Göschen, 1900. 8vo. 15 + 477 pp. M. 10.00

KELLER (C.). Algebra problems for use in Regents' and other high schools. Syracuse, N. Y., Bardeen, 1900. 16mo. 100 pp. Cloth. $0.50

LAZZERI (G.). Manuale di trigonometria sferica. Livorno, Giusti, 1900. 16mo. 95 pp. (Biblioteca degli studenti ; riassunti per tutte le materie d'esame nei licei, ginnasi, istituti tecnici, ecc., vols. 52. 53.) Fr. 1.00

LIGOWSKI (W.). Sammlung fünfstelliger logarithmischer, trigonometrischer und nautischer Tafeln, nebst Erklärungen und Formeln der Astronomie. 4te Auflage. Kiel, Universitätsbuchhandlung, 1900. 8vo. 23 + 212 + 48 pp. Cloth. M. 8.00

MILNER (F.). On teaching geometry. Boston, Heath, 1900. 16mo. 18 pp. (Heath's mathematical monographs, issued under the general editorship of W. Wells, No. 5.) $0.10

MONNIN (T.). L'année du certificat d'études. Livret d'arithmétique (opuscule du maître). (Réponses des problèmes du livret de l'élève ; autres problèmes analogues : partie complémentaire.) Paris, Colin, 1901. 16mo. 40 pp. Fr. 0.30

ORELLI (J.). Lehrbuch der Algebra für Industrie- und Gewerbeschulen, sowie zum Selbstunterricht. 3te Auflage in 2 Teilen. 3ter Abdruck. Vol. I : 8 + 304 pp. Vol. II : 7 + 286 pp. Zürich, Schmidt, 1901. 8vo. M. 10.00

REBIÈRE (A.). See BOS (H.).

RICHARDSON (G.). See WINCHESTER.

RÖLLNER (F.). Ueber Aehnlichkeit und Symmetrie als grundlegende Principien der Geometrie. Teil II. Römerstadt, 1900. 8vo. 28 pp.

ROHR (M. VON). Die Logarithmen der Sinus und Tangenten für 0° bis 5° und der Cosinus und Cotangenten für 85° bis 90° von tausendstel zu tausendstel Grad. Als Ergänzung zu C. Bremiker's fünfstelligen

Logarithmentafeln herausgegeben. Berlin, Weidmann, 1900. 8vo. 20 pp. M. $0.60

TESTI (G. M.). Elementi d'algebra, ad uso specialmente dei licenziandi delle scuole tecniche. 2a edizione, di nuovo modificata. Livorno, Giusti, 1900. 16mo. 7 + 86 pp. Fr. 1.20

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——. Teil II. Lehraufgabe der Obersekunda und der Prima des Gymnasiums. Breslau, Hirt, 1900. 8vo. Pp. 193–300. M. 1.25

WINCHESTER COLLEGE EXAMPLES in arithmetic, algebra and geometry; questions set in scholarship, Duncan prize, and other examinations. Edited by G. Richardson. London, Simpkin, 1901. 12mo. 312 pp. Cloth. 5 s.

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BOURLET (C.). See BRISSE (C.).

BRISSE (C.). Cours de géométrie descriptive (première partie), à l'usage des élèves de la classe de mathématiques élémentaires. 2e édition, revue par C. Bourlet. Paris, Gauthier-Villars, 1900. 8vo. 18 + 177 pp. Fr. 5.00

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GALILEI (G.). Opere. Edizione nazionale sotto gli auspici di S. M. il Re d'Italia. Volume X. Firenze, 1900. 4to. 531 pp.

HAUSSNER (R.). See MONGE (G.).

KHANDRIKOV (M.). Theory of the figure of the earth; higher geodesy. Kiev, 1900. 8vo. 222 pp. (Russian.) R. 2.00

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MONGE (G.). Darstellende Geometrie (1798). Uebersetzt und herausgegeben von R. Haussner. Leipzig, Engelmann, 1900. 12mo. 217 pp. (Ostwald's Klassiker der exakten Wissenschaften, No. 117.) Boards. M. 4.00

MONTANARI (C.). Elementi di geometria descrittiva, ad uso degli istituti tecnici. Livorno, Giusti, 1900. 16mo. 40 pp. (Biblioteca degli studenti; riassunti per tutte le materie d'esame nei licei, ginnasi, istituti tecnici, ecc., vol. 48.) Fr. 0.50

ROUGET (C.). Théorie des observations circumzénithales Mémoire présenté à l'Académie des sciences le 20 novembre, 1897. Paris, Gauthier-Villars, 1900. 8vo. 34 pp.

VALENTINER (S.). Untersuchungen über die Beziehung zwischen dem Potential einer homogenen Kugel und dem des Mittelpunktes. (Diss.) Karlsruhe, Braun, 1900. 8vo. 65 pp. M. 1.60

VECCHI (S.). Geometria descrittiva. Lezioni [dettate nella] r. università di Parma nell'anno 1899–1900, e compilate per cura di E. Beggi. Disp. 47–82. Parma, Zafferri, 1900. 8vo. Pp. 33–320.